3. (Quasi)-projective and general algebraic varieties

Affine varieties usually "go to infinity" when we draw them, this leads to complications in the theory.

Emple: Two distinct lines in $\mathbb{A}^2$ intersect in 1 point unless they are parallel, in which case they intersect "at ∞".

↳ as we will see, two lines in $\mathbb{P}^2$ always intersect in one point.

3.1 Projective space:

Def: Projective n-space $\mathbb{P}^n$ is the set

$\mathbb{P}^n = k^{n+1} \setminus \{0\} / \sim$, where $(x_1,\dots,x_{n+1}) \sim (y_1,\dots,y_{n+1})$ if $\exists \lambda \in k^*$ s.t. $x_i = \lambda y_i \ \forall i$.

Elements in $\mathbb{P}^n$ are called points. If $P \in \mathbb{P}^n$ is the equivalence class of $(x_1,\dots,x_{n+1}) \in \mathbb{A}^{n+1}$ we write $P = [x_1 : x_2 : \dots : x_{n+1}]$ and $x_1,\dots,x_{n+1}$ are called homogenous coordinates of P.

Rmk: Any point in $k^{n+1} \setminus \{0\}$ determines a line in $\mathbb{A}^{n+1}$ through the origin and $x, y \in k^{n+1} \setminus \{0\}$ define the same line $\Leftrightarrow y = \lambda \cdot x$ for some $\lambda \in k^*$. Hence $\mathbb{P}^n$ is naturally the set of lines in $\mathbb{A}^{n+1}$.

While the i-th coordinate x_i $(1 \le i \le n+1)$ of a point $[x_1 : \dots : x_{n+1}] \in \mathbb{P}^n$ is not well-defined, $x_i = 0$ or $x_i \neq 0$ is well-defined. We set

$$U_i := \{[x_1 : \dots : x_{n+1}] \in \mathbb{P}^n \mid x_i \neq 0\}.$$

Clearly $\mathbb{P}^n = \bigcup_{i=1}^{n+1} U_i$, since at least one coordinate has to be $\neq 0$.

Furthermore we have for every $1 \le i \le n+1$ a bijection

$$\varphi_i : \mathbb{A}^n \to U_i$$
$$(x_1,\dots,x_n) \mapsto [x_1 : \dots x_{i-1} : 1 : x_i : \dots : x_n]$$

with inverse $\left(\frac{x_1}{x_i}, \dots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \dots\right) \leftarrow [x_1 : \dots : x_{n+1}]$

Will see in a bit that φ_i's provide an open covering of $\mathbb{P}^n$

Will see in a bit that φ_i's provide an open covering of $\mathbb{P}$ by affine spaces.

Def: The set $H_\infty := \mathbb{P}^n \setminus U_{n+1} = \{[x_1:\dots:x_{n+1}] \in \mathbb{P}^n \mid x_{n+1} = 0\}$ is called the hyperplane at ∞. It can be identified with $\mathbb{P}^{n-1}$ and thus $\mathbb{P}^n = U_{n+1} \sqcup H_\infty$. "$(= \mathbb{A}^n \cup \mathbb{P}^{n-1})$"

$\mathbb{P}^1(\mathbb{C}) \cong$ 

Exples: · $\mathbb{P}^0 = k\setminus\{0\} / k^\times$ = pt.

· $\mathbb{P}^1 = U_2 \sqcup \mathbb{P}^0$ = Affine line $\sqcup$ pt at ∞ is called the projective line

· $\mathbb{P}^2 = \mathbb{A}^2 \sqcup \mathbb{P}^1$ is called the projective plane.

You'll see that two parallel lines in $\mathbb{A}^2$, when considered in $\mathbb{P}^2$ will intersect in $\mathbb{P}^1 = H_\infty$. (exercise)

3.2. Projective algebraic sets

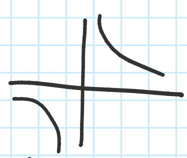
For a poly. $F \in k[X_1, \dots, X_{n+1}]$, the equation $F = 0$ in $\mathbb{P}^n$ is not well-def in general, it is however if F is a form of degree d since in that case $F(\lambda X_1, \dots, \lambda X_{n+1}) = \lambda^d F(X_1, \dots, X_{n+1})$ $\forall \lambda \in k^\times$.

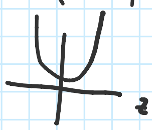
Def: · For any set $S \subset k[X_1, \dots, X_{n+1}]$ of homog. polynomials we set
$$V(S) := \{[x_1:\dots:x_{n+1}] \in \mathbb{P}^n \mid F(x_1, \dots, x_{n+1}) = 0 \ \forall F \in S\}$$

· A subset $V \subset \mathbb{P}^n$ is algebraic if it is of the form $V = V(S)$ w/ S as above.

Exple: $V = V(x^2 - yz) \subset \mathbb{P}^2$ How to picture it?

We can draw $V \cap U_i \subset U_i = \mathbb{A}^2$ (over $\mathbb{R}$):

$V \cap U_1 = V(1 - yz) \subset \mathbb{A}^2$ 

$V \cap U_2 = V(x^2 - z)$  same for $V \cap U_3$

This glues together to a curve in $\mathbb{P}^2(\mathbb{R})$ "circle"

As in the affine case, we're more interested in the ideal generated by S, rather than S itself.

<u>Def</u>: An ideal $I \subset k[x_1,\dots,x_n]$ is <u>homogeneous</u> if it's generated by homogeneous elements.

- For $I \subset k[x_1,\dots,x_{n+1}]$ homog. we set $V(I) := V(T) \subset \mathbb{P}^n$ w/ $T \subset I$ the set of forms in I.

<u>Rmk</u>: Since $k[x_1,\dots,x_{n+1}]$ is Noetherian, we can always pick a finite number of homog. generators for I.

- For $I = (x_1,\dots,x_{n+1})$ we have $V(I) = \emptyset$. We denote this ideal by $I_+ \subset k[x_1 \dots x_{n+1}]$, it's called the <u>irrelevant ideal</u>.

<u>Exple</u>: (x, y^2) is homog., $(x+y^2, x)$ is homog. but $(x+y^2)$ is not.

<u>Lemma 3.1</u>: $I \subset k[x_1 \dots x_n]$ is homog. if and only if for every $F \in I$, if we write $F = \sum_{i \geq 0} F_i$ w/ F_i homog. of degree i, we have $F_i \in I$ for all i.

<u>Pf</u>: $\Rightarrow$ Let $G^{(1)},\dots,G^{(k)}$ be a homog. set of generators for I w/ degrees $d_1,\dots,d_k$. Any $F = \sum_{i \geq 0} F_i \in I$ can be written as $F = \sum_{i=1}^{k} A^{(i)} G^{(i)}$, $A^{(i)} \in k[x_1,\dots,x_n]$. Since the degree is additive, we get $F_j = \sum_{i=1}^{k} A^{(i)}_{j-d_i} G^{(i)} \in I$ ✓.

<u>$\Leftarrow$</u>: Let $G^{(1)},\dots,G^{(k)}$ be any set of generators for I.
$\leadsto \forall j$, $G^{(i)}_j \in I \leadsto \{G^{(i)}_j\}_{\substack{j \geq 0 \\ 1 \leq i \leq k}}$ is a set of homog. generators for I □

<u>Further properties</u>: Sum, product, intersection and radical

Further properties: Sum, product, intersection and radical of homog. ideals are homog. A homog. ideal $I \subset k[X_1..X_n]$ is prime if for any homog. $f, g \in k[X_1, .., X_n]$

$$f \cdot g \in I \Rightarrow f \in I \text{ or } g \in I. \quad \text{(Exercise)}$$

Def: We define the Zariski topology on $\mathbb{P}^n$ by taking the open sets to be the complements of algebraic sets. (This is a topology by the further properties)

- An alg set $V \subset \mathbb{P}^n$ is irreducible if it's irred. as a topological space.

As in the affine case, there is a correspondence

$$\left\{\begin{matrix}\text{Alg subsets}\\ \text{of } \mathbb{P}^n\end{matrix}\right\} \underset{I}{\overset{V}{\rightleftarrows}} \left\{\begin{matrix}\text{Homog ideals}\\ \text{in } k[X_1,..,X_{n+1}]\end{matrix}\right\}$$

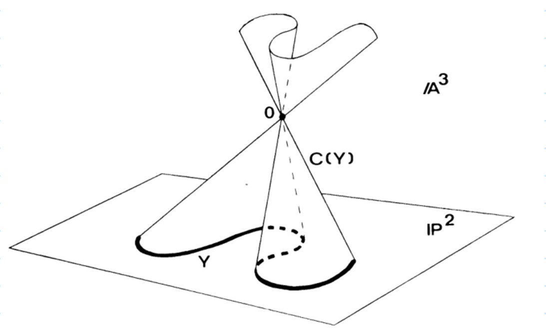
where $I(V) :=$ Ideal generated by $\left\{ F \in k[X_1,..,X_{n+1}] \;\middle|\; \begin{matrix}F \text{ homog.}\\ F(v)=0\\ \forall v \in V\end{matrix} \right\}$

Rmk: If we need to distinguish between the affine and the projective correspondence we write V_a, I_a and V_p, I_p respectively. The two are not independent:

Def: For $V \subset \mathbb{P}^n$ alg. we define the cone over V as

$$C(V) = \{ (x_1,..,x_{n+1}) \in \mathbb{A}^{n+1} \mid [x_1 : .. : x_{n+1}] \in V \} \cup \{0,..,0\} \subset \mathbb{A}^{n+1}$$

Expl: $V = \{x\} \subset \mathbb{P}^n$, then $C(V)$ is the line corresp. to x.



Cone over a curve Y in $\mathbb{P}^2$.

Lemma 3.2 1) If $V \neq \emptyset$ then $I_p(V) = I_a(C(V))$

2) If $I \subsetneq k[x_1, \dots, x_{n+1}]$ is homog., then

$$C(V_p(I)) = V_a(I)$$

pf: 1) $G \in I_p(V)$ homog. and $(x_1, \dots, x_{n+1}) \in C(V) \setminus \{0\}$. Then

$G(x_1, \dots, x_{n+1}) = 0 \rightsquigarrow G \in I_a(C(V) \setminus \{0\})$ G homog $\Rightarrow G \equiv 0$ on $\deg G \geq 1$ so also $G(0) = 0$ $\Rightarrow G \in I_a(C(V))$.

Conversely if $G \in I_a(C(V))$, write $G = \sum_{i=0}^{d} G_i$

w/ G_i homog. of deg i.

For every $x \in C(V)$ and $\lambda \in k^\times$ we have $\lambda \cdot x \in C(V)$

$$\rightsquigarrow 0 = G(\lambda \cdot x) = \sum_{i=0}^{d} \lambda^i G_i(x)$$

$\rightsquigarrow \tilde{G}(Y) := \sum Y^i G_i(x) \in k[Y]$ has ∞-many 0's $\rightsquigarrow \tilde{G} = 0$

$\Leftrightarrow G_i(x) = 0 \ \forall 1 \leq i \leq d \rightsquigarrow G_i \in I_p(V)$. $\rightsquigarrow G \in I_p(V)$ $\overset{1)}{\square}$

2) $C(V_p(I)) = \{ x \in \mathbb{A}^{n+1} \mid g(x) = 0 \ \forall g \in I \text{ homog.} \}$

$= \{ x \in \mathbb{A}^{n+1} \mid g(x) = 0 \ \forall g \in I \} = V_a(I)$. $\square$

As in the affine case, there is a Nullstellensatz, which makes the corresp (V_p, I_p) more precise:

Proposition 3.3 (Projective Nullstellensatz): Let $I \subset k[x_1, \dots x_{n+1}]$ be a homog. ideal. Then

1) $V_p(I) = \emptyset \Leftrightarrow I = k[x_1, \dots, x_{n+1}]$ or $\mathrm{Rad}(I) = I_+$

2) If $V_p(I) \neq \emptyset$ then $I_p(V_p(I)) = \mathrm{Rad}(I)$

pf: 1)

$$V_p(I) = \emptyset \overset{3.2}{\Leftrightarrow} \underset{I = k[x_1, \dots, x_{n+1}]}{V_a(I) = \{(0, \dots, 0)\}} \Leftrightarrow \mathrm{Rad}(I) \overset{\text{usual NSS}}{=} I_a(V_a(I)) = I_+$$

2) $I_p(V_p(I)) \overset{3.2}{=} I_a(C(V_p(I))) \overset{3.2}{=} I_a(V_a(I)) = \mathrm{Rad}(I)$.

2) $I_p(V_p(I)) \overset{3.2}{=} I_a(C(V_p(I))) \overset{3.2}{=} I_a(V_a(I)) = \mathrm{Rad}(I)$. $\square$

<u>Corollary 3.4</u>: There is a 1-1 inclusion-reversing corresp.

$$\left\{\begin{array}{l}\text{radical homog.}\\ \text{ideals in } k[X_1,\dots,X_{n+1}]\\ \text{different from } I_+\end{array}\right\} \underset{I_P}{\overset{V_P}{\rightleftarrows}} \left\{\begin{array}{l}\text{alg. subsets}\\ \text{of } \mathbb{P}^n\end{array}\right\}.$$

s.t. $V_P(I)$ is irred. $\Leftrightarrow$ I is prime.

<u>pf</u>: As in the affine case $\square$

<u>Rmk</u>: Notice however that points in $\mathbb{P}^n$ do not corresp. to maximal ideals: $x \in \mathbb{P}^n$, $I_p(x) \overset{3.2}{=} I_a(C(x)) \subset k[X_1,\dots X_{n+1}]$

a line.

We can also relate affine and proj. alg sets through

$$\varphi_i : \mathbb{A}^n \to U_i = \{x_i \neq 0\} \subset \mathbb{P}^n.$$

We focus on $\varphi = \varphi_{n+1} : \mathbb{A}^n \to U = U_{n+1}$.

For $F \in k[X_1,\dots X_{n+1}]$ homog. we define

$$F_*(X_1,\dots,X_{n+1}) := F(X_1,\dots X_n, 1) \in k[X_1,\dots,X_n]$$

Conversely for $G \in k[X_1,\dots,X_n]$ we write $G = G_0 + \dots + G_d$ w/ G_i a form of deg i, and set

$$G^*(X_1,\dots X_{n+1}) = X_{n+1}^d \cdot G_0 + X_{n+1}^{d-1} G_1 + \dots + G_d$$
$$= X_{n+1}^d \, G\left(\frac{X_1}{X_{n+1}},\dots,\frac{X_n}{X_{n+1}}\right)$$

Then G^* is a form of deg d in $k[X_1,\dots,X_{n+1}]$

<u>Def</u>: $(\cdot)_*$ and $(\cdot)^*$ are called <u>dehomogenization</u> and <u>homogenisation</u> respectively. (with respect to X_{n+1}).

- For $I \subset k[X_1,\dots,X_n]$ let I^* be the homog. ideal generated by $\{F^* \mid F \in I\}$. If $V = V_a(I) \subset \mathbb{A}^n$ set $V^* := V_p(I^*) \subset \mathbb{P}^n$.

V^* is called the proj. closure of V

by $\{F^* \mid F \in I\}$. If $V = V_a(I) \subset \mathbb{A}^n$ set $V^* := V_p(I^*) \subset \mathbb{P}^n$.

V^* is called the <u>proj. closure</u> of V

- Similarly for $I \subset k[X_1, \dots, X_{n+1}]$ homog., $I_* = \{F_* \mid F \in I\}$ is an ideal and if $V = V_p(I)$ we set $V_* = V_a(I_*)$.

<u>Exple:</u> $F = X_1^2 - X_2 \in k[X_1, X_2]$. Then $F^* = X_1^2 - X_2 \cdot X_3$.

$\rightsquigarrow$ proj closure of $V_a(X_1^2 - X_2) \subset \mathbb{A}^2$ is $V_p(X_1^2 - X_2 X_3) \subset \mathbb{P}^2$.

<u>Lemma 3.5:</u> If $V \subset \mathbb{A}^n$ is closed, then $\varphi(V) = V^* \cap U \subset \mathbb{P}^n$

If $V \subset \mathbb{P}^n$ is closed then $\varphi^{-1}(V \cap U) = V_*$

In particular $\varphi : \mathbb{A}^n \to U$ is a homeomorphism.

<u>Pf:</u> Recall that $\varphi(x) = [x_1 : \dots : x_n : 1]$ for $x \in \mathbb{A}^n$

For $V \subset \mathbb{A}^n$, ~~write~~ let $I = I(V)$ and

$V^* = V_p(I^*) = V_p(\{F^* \mid F \in I\})$

But $F^*(X_1, \dots, X_{n+1}) = X_{n+1}^{\deg(F)} F\left(\frac{X_1}{X_{n+1}}, \dots, \frac{X_n}{X_{n+1}}\right)$

$\rightsquigarrow F(v) \overset{?}{=} 0 \iff F^*(\varphi(v)) = 0 \rightsquigarrow \varphi(V) = V^* \cap U$

$\varphi^{-1}(V \cap U) = V_*$ is proven similarly.

We see that both φ, φ^{-1} send closed sets to closed sets, thus they are continuous $\rightsquigarrow \varphi, \varphi^{-1}$ are homeom. $\square$

<u>3.3 (quasi-)projective and general varieties:</u>

<u>Def:</u> A <u>projective variety</u> is an irreducible algebraic subset of $\mathbb{P}^n$.

- A <u>quasi-projective variety</u> is an open subset of a proj. var.
- An <u>algebraic variety</u> is an affine, quasi-affine, proj. or quasi-proj variety.

quasi-proj variety.

Rmk: In order to define morphisms between varieties, we still need to define the ring $\mathcal{O}(V)$ of reg. fcts on a (quasi)-proj. V.

If $V = V(I) \subset \mathbb{P}^n$ is proj., a natural guess would be
$$\mathcal{O}(V) = k[x_1, \dots, x_{n+1}]/I$$
but elements in $k[x_1, \dots, x_{n+1}]$ don't define fcts on $\mathbb{P}^n$!

Def: Let $V \subseteq \mathbb{P}^n$ be quasi-proj. A map $f: V \to k$ is regular at $p \in V$, if $\exists$ open nbhd $U \subseteq V$ of p and forms $g, h \in k[x_1, \dots, x_{n+1}]$ of the same degree s.t. $h(u) \neq 0$ $\forall u \in U$ and $f(u) = \frac{g(u)}{h(u)}$ $\forall u \in U$.

Rmk: $\frac{g(a)}{h(a)} \in k$ is well-def. since g and h have the same degree.

We write $\mathcal{O}(V)$ for the ring of regular functions on V.

Rmk: As in the quasi-affine case, regular fcts are continuous when $k = \mathbb{A}^1$ is given the Zariski-top.

- If $f: V \to k$ is regular and $f(v) \neq 0$ $\forall v \in V$, then also $1/f$ is regular. $\rightsquigarrow$ $\mathcal{O}(V)^\times = \{ f \mid f(v) \neq 0 \ \forall v \in V \}$.
- Given $f: \mathbb{P}^1 \to k$ regular. Write $\mathbb{P}^1 = \mathbb{A}^1 \cup \infty$. Then $f|_{\mathbb{A}^1}: \mathbb{A}^1 \to k$ $\Rightarrow$ $f \in k[x]$. But if $\deg f \geq 1$, $f(\infty)$ cannot be defined.

This suggests $\mathcal{O}(\mathbb{P}^1) \cong k$ $\leftarrow$ constant fcts. (Exercise)

In fact, (without proof)

Thm: V proj. variety, then $\mathcal{O}(V) \cong k$.

(c.f. Liouville thm in cplx analysis)

3.4 Morphisms of varieties

3.4 Morphisms of varieties

<u>Def:</u> A <u>morphism</u> between two varieties V, W is a continous map
$$\varphi : V \longrightarrow W$$
s.t. for every open $U \subset W$ and every $f \in \mathcal{O}(U)$ the map $f \circ \varphi : \varphi^{-1}(U) \longrightarrow k$ is regular.

- φ is an <u>isomorphism</u> if $\exists \psi : W \to V$ s.t. $\varphi \circ \psi = 1_W, \psi \circ \varphi = 1_V$.

<u>Rmk:</u> In part. any $\varphi : V \to W$ induces $\tilde{\varphi} : \mathcal{O}(W) \longrightarrow \mathcal{O}(V)$
The converse is only true if W is affine (see below)

<u>Proposition 3.6:</u> . The maps $\varphi_i : \mathbb{A}^n \to U_i \subset \mathbb{P}^n$ are isomorphisms
In fact, for every quasi-affine $V \subset \mathbb{A}^n$, $\varphi_{i|V}$ is an isom. onto its image $\varphi_i(V)$, which is quasi-proj.

<u>pf</u> We may take $i = n+1$. and write $\varphi = \varphi_{n+1} : \mathbb{A}^n \to U$.
We already saw that φ is a homeo. $\leadsto \varphi(V)$ is quasi-proj:
$$V \subset \overline{V} \subset \mathbb{A}^n \leadsto \varphi(V) \subset \varphi(\overline{V}) \underset{3.5}{=} \overline{V}^* \cap U \subset \overline{V}^* \subset \text{proj}$$

Let f be a regular function on some open $W \subset \varphi(V)$. By shrinking W if necessary $f = \frac{G}{H}$ w/ G, H forms of the same degree and $H(w) \neq 0 \ \forall w \in W$.
Then $F \circ \varphi = \frac{G \circ \varphi}{H \circ \varphi} = \frac{G_*}{H_*}$ is regular on $\varphi^{-1}(W)$.
$\leadsto \varphi$ is a morphism of varieties.
Conversly we have $\varphi^{-1} : U \to \mathbb{A}^n$.

Let $W \subset V$ and $f \in \mathcal{O}(W)$ regular, i.e. up to shrinking $f = \frac{G}{H}$.

Then $F \circ \varphi^{-1}([X_1 : \ldots : X_{n+1}]) = \dfrac{G\left(\frac{X_1}{X_{n+1}}, \ldots, \frac{X_n}{X_{n+1}}\right)}{H(\quad '' \quad)} = X_{n+1}^{\alpha} \dfrac{G^*([X_1 : \ldots : X_{n+1}])}{H^*([X_1 : \ldots : X_{n+1}])}$

for a unique $\alpha \in \mathbb{Z}$.

If $\alpha \geq 0$ then $X_{n+1}^{\alpha} G^*$ and H^* are homog. of the same deg.

If $\alpha \leq 0$ then G^* and $X_{n+1}^{-\alpha} H^*$ ''

$\rightsquigarrow F \circ \varphi^{-1} \in \mathcal{O}\left((\varphi^{-1})^{-1}(W)\right) \rightsquigarrow \varphi^{-1}$ is a morphism $\square$

<u>Def</u>: A variety V is <u>(quasi-)affine</u> or <u>(quasi-)projective,</u>

if V is isom to a '' '' variety.

<u>Corollary 3.7</u>: 1) Any variety is quasi-projective.

2) Every (quasi)-proj. var. $V \subset \mathbb{P}^n$ admits a finite open cover by (quasi)-affine varieties, namely $V = \bigcup_{i=1}^{n+1} V \cap U_i$.

Pf 1) Clearly proj $\Rightarrow$ quasi-proj, affine $\Rightarrow$ quasi-affine and 3.6 implies quasi-affine $\Rightarrow$ quasi-proj $\square$

2) Since $\mathbb{P}^n = \bigcup_{i=1}^{n+1} U_i$, $V = \bigcup_{i=1}^{n+1} V \cap U_i$ $\qquad V \cap U_i \subset V$ is open.

If $V \subset \mathbb{P}^n$ is proj., then $V \cap U_i$ is closed in $U_i \cong \mathbb{A}^n$ and irred

$\rightsquigarrow V \cap U_i$ is an aff. var.

If $V \subset \bar{V} \subset \mathbb{P}^n$ quasi-proj. then $V \cap U_i$ is quasi-aff. in $\bar{V} \cap U_i$. $\square$

<u>Expl</u>/Exercise: · For any $f \in k[X_1, \ldots, X_n]$ $\mathbb{A}^n \setminus V(f)$ is affine

· $\mathbb{A}^2 \setminus \{0\} = \mathbb{A}^2 \setminus V(x, y)$ is not affine

· $\mathbb{A}^2 \setminus \{0\} = \mathbb{A}^2 \setminus V(x,y)$ is not affine

Rmk: 3.6 also shows that if V is quasi-affine, then $\mathcal{O}(V) \cong \mathcal{O}(\varphi(V))$, quasi-proj $\rightsquigarrow$ all definitions of $\mathcal{O}(-)$ are compatible with each other.

The definition of a morphism is clean, but difficult to apply in practice. It is somewhat simpler if at least W is affine.

Lemma 3.8: Let $\varphi: V \to W$ be any map, $W \subset \mathbb{A}^n$ affine and $x_1, \dots, x_n: \mathbb{A}^n \to \mathbb{A}^1$ the coordinate functions. Then φ is a morphism if and only if $x_i \circ \varphi: V \to \mathbb{A}^1$ is regular for $1 \leq i \leq n$.

pf: If φ is a morph, then $x_i \circ \varphi$ is regular by def.
Conversely if $x_i \circ \varphi$ is regular for $1 \leq i \leq n$, then also $f \circ \varphi$ for any $f \in k[x_1, \dots, x_n]$, since regular fct's form a ring.
(e.g. $(x_i \circ \varphi)^n = x_i^n \circ \varphi$ ...)

$\rightsquigarrow \varphi^{-1}(V(f_1, \dots, f_k)) = \varphi^{-1}(\bigcap V(f_i)) = \bigcap (f_i \circ \varphi)^{-1}(0)$ is closed

$\rightsquigarrow \varphi$ is continous.

Now let $U \subset W$ be open and $f \in \mathcal{O}(U)$. By maybe shrinking U we may assume $f = \frac{g}{h}$ w/ $g, h \in k[x_1, \dots, x_n]$, $h(u) \neq 0$ $\forall u \in U$.

$$f \circ \varphi = \frac{g \circ \varphi}{h \circ \varphi}$$ and $g \circ \varphi$, $h \circ \varphi$ are regular and $h \circ \varphi(v) \neq 0$ $\forall v \in \varphi^{-1}(U)$. $\rightsquigarrow \frac{1}{h \circ \varphi}$ is also regular.

$\rightsquigarrow \frac{g \circ \varphi}{h \circ \varphi} = f \circ \varphi$ is regular $\square$

Exple: $\mathbb{A}^1 \setminus \{0\} \longrightarrow V(xy-1) \subset \mathbb{A}^2$ is a morphism,

Exple: $\mathbb{A}^1 \setminus \{0\} \longrightarrow V(XY-1) \subset \mathbb{A}^2$ is a morphism,
$x \mapsto (x, \frac{1}{x})$

since both x, and $\frac{1}{x}$ are regular fct's on $\mathbb{A}^1 \setminus \{0\}$.
The morph. $\mathbb{A}^2 \longrightarrow \mathbb{A}^1$, $(x,y) \mapsto x$ provides an inverse, when restricted to $V(XY-1)$

$\rightsquigarrow V(xy-1) \simeq \mathbb{A}^1 \setminus \{0\}$

Lemma 3.8 allows us to understand morphisms to affine varieties in general:

Corollary 3.9: Let V, W be varieties with W affine. There is a bijection $\mathrm{Hom}_{var}(V, W) = \mathrm{Hom}_{k\text{-}alg}(\mathcal{O}(W), \mathcal{O}(V))$

pf: Exercise (similar to Prop 2.5).

3.5 General rational functions and local rings

Let V be an algebraic variety and $P \in V$.

Def: The local ring $\mathcal{O}_P(V)$ of V at P is the set of pairs (U, f), where $U \subset V$ is open containing P and f is a regular function on U, modulo the following equivalence relation:

$$(U, f) \sim (U', f') \iff f = f' \text{ on } U \cap U'.$$

Lemma 3.10: 1) $\sim$ is an equivalence relation and $\mathcal{O}_P(V)$ is a ring with the operations

$$[U, f] + [U', f'] := [U \cap U', f + f']$$
$$[U, f] \cdot [U', f'] := [U \cap U', f \cdot f'].$$

2) $\mathcal{O}_P(V)$ is a local ring with maximal ideal

$= \{[U, f] \mid f(P) = 0\}$

2) $\mathcal{O}_P(V)$ is a local ring with maximal ideal

$$\mathfrak{m}_P = \{ [U,f] \mid f(P) = 0 \}.$$

pf: 1) We need to check transitivity for $\sim$:

If $(U,f) \sim (U',f')$ and $(U',f') \sim (U'',f'')$, then

$f = f''$ on $U \cap U' \cap U''$. But $U \cap U' \cap U''$ is open and dense in $U \cap U''$ and f, f'' are continous

$\rightsquigarrow f = f''$ on $U \cap U''$ $\Leftrightarrow (U,f) \sim (U'', f'')$

2) We have an evalution morphism

$$\mathcal{O}_P(V) \xrightarrow{ev} K$$
$$[U,f] \longmapsto f(P)$$

It is surjective since constant fcts are in $\mathcal{O}_P(V)$.

$\rightsquigarrow \mathfrak{m}_P = \ker(ev)$ is maximal.

Finally $\mathcal{O}_P(V)$ is local since $[U,f] \notin \mathfrak{m}_P \Leftrightarrow f(P) \neq 0$

$\Leftrightarrow \frac{1}{f}$ is regular in a nbhd of P.

$\rightsquigarrow [U,f] \in \mathcal{O}_P(V)^\times$. $\square$

Def: The field of rational functions $K(V)$ is the set of pairs (U,f) with $U \subset V$ open non-empty and f a regular function on U modulo the equiv rel. $(U,f) \sim (U',f') \Leftrightarrow f = f'$ on $U \cap U'$.

Rmk: Since V is irreducible, any non-empty open $U \subset V$ is dense. Hence $U \cap U'$ is non-empty open aswell. $\rightsquigarrow$ As in Lemma 3.10 we see that $\sim$ is an equiv. relation and $+, \cdot$ are well def.

For any $[U,f]$, $\frac{1}{f}$ is regular on the open $U' = U \setminus f(0)$

For any $[U,f]$, $\frac{1}{f}$ is regular on the open $U' = U \setminus f(0)$

$\rightsquigarrow [U', \frac{1}{f}]$ is inverse to $[U,f]$ if $f \neq 0 \rightsquigarrow k(V)$ is a field.

- As in the affine case we have inclusions

$$\begin{array}{ccccc} \mathcal{O}(V) & \hookrightarrow & \mathcal{O}_p(V) & \hookrightarrow & k(V) \quad \text{for any } p \in V. \\ f & \mapsto & [V,f] & & \\ & & [U,g] & \longmapsto & [U,g] \end{array}$$

<u>Proposition 3.11:</u>

1) Let V be an alg. variety and $U \subset V$ open non-empty. Then $k(V) = k(U)$ and $\mathcal{O}_p(V) = \mathcal{O}_p(U)$ for any $p \in U$.

2) For any alg var. V and any $p \in V$, $k(V)$ is the quotient field of $\mathcal{O}_p(V)$

3) If V is affine then $\mathcal{O}_p(V) = \Gamma(V)_{m_p}$ for any $p \in V$ and $k(V)$ is the quotient field of $\Gamma(V)$

In particular all definitions for $\mathcal{O}_p(V)$ and $k(V)$ agree for affine varieties.

<u>Rmk:</u> 1) implies for $V \subset \mathbb{P}^n$ quasi-proj and $p \in V \cap U_i$ for some i, that $\mathcal{O}_p(V) = \mathcal{O}_p(\bar{V}) = \mathcal{O}_p(\bar{V} \cap U_i)$

Since $U_i \simeq \mathbb{A}^n$, $\bar{V} \cap U_i$ is affine and $\mathcal{O}_p(\bar{V} \cap U_i) = \Gamma(\bar{V} \cap U_i)_{m_p}$.

Hence in principle we can always express $\mathcal{O}_p(V)$ as an explicit localization.

<u>pf:</u> 1) Immediate from the definition: $[W,f] \in \mathcal{O}_p(V)$

$$\mathcal{O}_p(U) \ni [W \cap U, f] \in \mathcal{O}_p(V).$$

$\mathcal{O}_p(\mathcal{U}) \ni [W \cap U, f] \in \mathcal{O}_p(V)$.

2) As in the Rmk, we can reduce to the affine case, and then it follows from 3).

3) We have a map
$$\Gamma(V)_{m_p} \longrightarrow \mathcal{O}_p(V)$$
$$f/g \longmapsto [V, f/g] \quad (g \notin m_p).$$

It is injective by continuity.

And also surj. by definition of $\mathcal{O}_p(V)$:

Any $[U, h] \in \mathcal{O}_p(V)$ is of the form $h = f/g$ w/ $g(u) \neq 0$ on U.

Since U is quasi-affine f, g are quotients of regular fcts on $V = \overline{U}$. $\leadsto$ the map is surjective.

Similarly we have $Q(\Gamma(V)) \hookrightarrow k(V)$
$$f/g \mapsto [V, f/g].$$

But any $[U, h] \in k(V)$ is contained in $\mathcal{O}_p(V)$ for any $p \in U$.
$\cong \Gamma(V)_{m_p} \subset Q(\Gamma(V))$

$\leadsto$ Get $k(V) \hookrightarrow Q(\Gamma(V))$

$\square$

3.6 Dimension of a Variety

The dimension of a variety should be a basic invariant, but it turns out to be quite tricky to compute.

__Def:__ The __dimension__ of a topological space X is the largest integer n s.t. there exists a chain $Z_n \supset Z_{n-1} \supset \ldots \supset Z_1 \supset Z_0$ of distinct irreducible closed subsets of X.

- The __dimension of an alg. set__ is its dimension as a topological space

The dimension of an alg. set. is its dimension as a topological space.

Of course we'd like to relate this definition with the dimension theory for rings.

Recall: · In any ring A, the height $ht(\mathfrak{p})$ of a prime ideal $\mathfrak{p} \subset A$ is the supremum of all integers n s.t. $\exists$ chain $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \dots \subset \mathfrak{p}_n = \mathfrak{p}$ of distinct prime ideals.

· The (Krull)-dimension of A is

$$\dim A = \sup \{ ht(\mathfrak{p}) \mid \mathfrak{p} \subset A \text{ prime} \}$$

Proposition 3.12: If $V \subset \mathbb{A}^n$ is an affine alg. variety then

$$\dim V = \dim \mathcal{O}(V) = \dim \Gamma(V)$$

Pf: If $V_0 \subset V_1 \subset \dots \subset V_d$ a max. chain of irred. closed subsets in V ($d < \infty$ by Noetherianity). Then $V_d = V$ and

$$I(V_0) \supset I(V_1) \supset \dots \supset I(V_d) = I(V)$$

is a chain of distinct prime ideals containing $I(V)$

$\rightsquigarrow$ Get a chain $\mathfrak{p}_0 \supset \mathfrak{p}_1 \supset \dots \supset \mathfrak{p}_d = (0)$ in $\mathcal{O}(V) = k[x_1, \dots, x_n]/I(V)$.

$\rightsquigarrow \dim \mathcal{O}(V) \geq \dim V$.

We can go the other way with $V(\cdot)$ $\square$

Actually computing dimensions is hard, the main tool is the following thm from commutative algebra:

Thm 3.13: Let k be a field and B a domain, which is a finitely generated k-algebra. Then

a)
$$\dim B = \text{transcendence degree of } Q(B)/k$$

b) For any prime ideal $\mathfrak{p} \subset B$ we have
$$\operatorname{ht}(\mathfrak{p}) + \dim(B/\mathfrak{p}) = \dim B$$

pf: Rings and modules.

Corollary 3.14: $\dim \mathbb{A}^n = n$

pf: We have $\dim \mathbb{A}^n \underset{3.12}{=} \dim k[X_1, \dots, X_n] \overset{a)}{=} \operatorname{tr-deg} k(X_1 \dots X_n)/k = n$ $\square$

Corollary 3.15 If $V \subset \mathbb{P}^n$ is a quasi-proj. variety, then
$$\dim \overline{V} = \dim V.$$
In part. $\dim \mathbb{P}^n = n$

We'll need the following

Lemma 3.16: If X is a top. space and $\{U_i\}$ a family of open subsets covering X. Then
$$\dim X = \sup_i \dim U_i$$

pf: Exercise.

pf of 3.15: Let $U_1, \dots, U_{n+1}$ be the open charts of $\mathbb{P}^n$.
Then $V_i = V \cap U_i$ gives an open covering of V by quasi-affine varieties and $\overline{V} \cap U_i = \overline{V_i}$ ~ closure in $U_i \cong \mathbb{A}^n$.
$\leadsto$ If we know 3.15 for quasi-affine varieties, we get
$$\dim \overline{V} \underset{3.16}{=} \sup_i \dim \overline{V_i} = \sup_i \dim V_i \underset{3.16}{=} \dim V.$$

$\dim V \underset{3.16}{=} \sup_i \dim V_i = \sup_i \dim \overline{V_i} \underset{3.16}{=} \dim \overline{V}$.

Now assume $V \subset \mathbb{A}^n$ is quasi-aff. of dim d and let $Z_0 \subset \ldots \subset Z_d$ be a max chain of irred. in V.

$\rightsquigarrow Z_d = V$ and $Z_0 = \{x\}$.

Let $\mathfrak{m} \subset \mathcal{O}(\overline{V})$ be the max. ideal corresp. to x.

<u>Claim</u> $\operatorname{ht}(\mathfrak{m}) = d$

If true, then we have

$$\dim \overline{V} \underset{3.12}{=} \dim \mathcal{O}(\overline{V}) \underset{3.13\, b)}{=} \dim(\underbrace{\mathcal{O}(\overline{V})/\mathfrak{m}}_{0}) + \operatorname{ht}(\mathfrak{m}) = d = \dim(V).$$

<u>pf of claim:</u> Clearly $\operatorname{ht}(\mathfrak{m}) \geq d$. Let $\mathfrak{P}_0 \subset \ldots \subset \mathfrak{P}_r = \mathfrak{m}$ be a chain of distinct prime ideals in $\mathcal{O}(\overline{V})$ w/ $r > d$.

Set $W_i = V(\mathfrak{P}_i) \rightsquigarrow \overline{V} = W_0 \supset \ldots \supset W_r = \{x\}$

Intersecting with V we get

$$V = W_0 \cap V \supset W_1 \cap V \ldots \supset W_r \cap V = \{x\} \neq \emptyset.$$

Since $W_i \cap V \subset W_i$ open non-empty $\rightarrow W_i \cap V$ is irred.

By max. of the initial seq. $\exists\, i$ s.t.

$$W_i \cap V = W_{i+1} \cap V$$

But then $W_i = \overline{W_i \cap V} = \overline{W_{i+1} \cap V} = W_{i+1}$ ↯ $\mathfrak{P}_i$'s were distinct! $\square$

Our intuition from linear alg. tells us, that $V \subset \mathbb{A}^n$ given by r "independent" equations should have dimension $n-r$.

The next proposition makes this precise:

<u>Proposition 3.17:</u>

Proposition 3.17:

a) Let $V \subset \mathbb{A}^n$ be an affine variety of dim d and $H = V(f) \subset \mathbb{A}^n$ a hypersurface s.t. $V \nsubseteq H$. Then every irred. comp. of $V \cap H$ has dimension $d-1$.

b) Let $I \subset k[X_1, .., X_n]$ be an ideal that can be generated by r polynomials. Then every irred comp. of $V(I)$ has dimension $\geq n-r$.

Rmk: One has a similar statement in the proj. case.

Warning: It's not true that if we choose the number of generators minimal in b), that we get an equality.

Expl: $I = (XY, YZ) \subset k[X,Y,Z]$

Then $V(I) = \underbrace{V(Y)}_{\text{dim } 2} \cup \underbrace{V(X,Z)}_{\text{dim } 1}$

It even fails when $V(I)$ is irred. (Exercise)

For the proof we'll need the following fact from comm. algebra, that we won't prove:

Thm 3.18 (Krull's Hauptidealsatz): Let A be a Noetherian ring and $f \in A$ neither a 0-divisor nor a unit. Then every minimal prime ideal containing f has height 1.

pf of prop: $V \nsubseteq H = V(f) \iff f \neq 0$ in $\mathcal{O}(V) \rightsquigarrow f$ not a 0-div.

pf of prop: $V \supsetneq H = V(f) \Leftrightarrow f \neq 0$ in $\mathcal{O}(V) \rightsquigarrow f$ not a 0-div.

$f \in \mathcal{O}(V)$ is a unit $\Leftrightarrow V \cap H = \emptyset \rightsquigarrow$ nothing to prove

Otherwise an irred. comp of $V \cap H$ corresponds to a prime ideal $\mathfrak{P}$ of $\mathcal{O}(V)$, in fact a minimal one, containing f.

$\overset{3.18}{\rightsquigarrow} \operatorname{ht} \mathfrak{P} = 1 \overset{3.13}{\rightsquigarrow} \dim(V(\mathfrak{P})) = \dim \mathcal{O}(V)/\mathfrak{P} = \dim \mathcal{O}(V) - \operatorname{ht}\mathfrak{P}$
$= d - 1$ □

b) By induction. $r = 0$ ✓

Let $I = (f_1, \dots, f_r)$, $r \geq 1$ and W for an irred. comp. of $V(I)$.

By induction any irred. comp. W' of $V(f_1, \dots, f_{r-1})$ has $\dim \geq n-(r-1)$.

By a) every irred comp. of $W' \cap V(f_r)$ has $\dim \geq n-(r-1)-1$
$= n - r$.

W is a union of irred. comp. of $W' \cap V(f_r)$ (possibly with varying W') $\rightsquigarrow \dim W \geq n - r$ □